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Application of hysteresis functions in vibration problems

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Abstract

In the paper, the notion of the hysteresis function introduced by the author is developed further and applied in some vibration problems for hysteretic materials. The stress–strain relation is built on the basis of well-known extended Masing's rules in which the scaled backbone curve used in the second rule is replaced by a more general function called the hysteresis function. This modification allows us to regulate the dependence of the damping ratio on the strain amplitude in the process of cycle deforming of a material with further extending to arbitrary deformation processes. The three cases of hysteretic systems are considered: the system with limited stress when strain increases without bound, the system with linear backbone curve, and the system with the backbone curve having increasing stiffness for increasing strain. For a number of dynamic examples, a comparison of different hysteresis functions is carried out.

1. Introduction

The well-known extended Masing's rules, defining relationships between stress τ and strain γ for a mechanical system with limited strength, are based on the behavior of a system comprising large (infinite) number of Prandtl's elements (joined in parallel) or the elements, each of which consists of a spring in parallel with a Coulomb's slider, joined in series. The above models are called in publications as Iwan's models [1,2]. In book by Palmov [3] one can find description of different elastic–plastic models along with approximate analytical solutions of some dynamic problems.

It seems appropriate to give here the formulation of the four extended Masing's rules [4]:

1. For initial loading, the stress–strain curve follows the backbone curve $\tau = F_{\mathrm{bb}}(\gamma)$.
2. If a stress reversal occurs at a point defined by $(\gamma_{\mathrm{rev}},\ \tau_{\mathrm{rev}})$, the stress–strain curve follows a path given by

$$\frac{\tau - \tau_{\mathrm{rev}}}{2} = F_{\mathrm{bb}}\left(\frac{\gamma - \gamma_{\mathrm{rev}}}{2}\right) \tag{1}$$

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3. If the unloading or reloading curve exceeds the maximum past strain and intersects the backbone curve, it follows the backbone curve until the next stress reversal.
4. If an unloading or reloading curve crosses an unloading or reloading curve from the previous cycle, the stress–strain curve follows that of the previous cycle.

Assume, e.g., that the backbone curve is a hyperbola with an initial modulus G_0 and limit stress τ_u:

$$y = \frac{x}{1 + |x|} \tag{2}$$

where x and y are the normalized strain and stress, respectively, which are defined by a strain γ and stress τ as follows:

$$x = \frac{\gamma}{\gamma_r}, \quad y = \frac{\tau}{\tau_u}, \quad \gamma_r = \frac{\tau_u}{G_0} \tag{3}$$

Such normalization is desired for any backbone curve with limited strength. Eq. (2) will be used below in examples.

The attempts to modify the Masing's rules have been made by Pyke [5], Archuleta et al. [6], Osinov [7]. Pyke [5] retains only the first rule and uses for subsequent loadings and unloadings, hyperbolic curves with an asymptotes defined by strength of the material. Advantages of such a treatment consist in simplicity of tracking the stress–strain relationship and in limitation of stresses by the strength of the considered material. However, the departure from Masing's rules results in the behavior which may be not consistent with that of the studied material (see Ref. [8]). The similar deviations from the behavior inherent in Iwan's type models or in models obeying the four extended Masing's rules are observed in other models, e.g., suggested by Osinov [7], Archuleta et al. [6], Bouc–Wen model [9,10].

One of the important characteristics of a model is the damping ratio corresponding to symmetric cyclic deforming of a material. Classical Masing's model as well as such Pyke's model, model of Archuleta et al. do not allow regulating this parameter which in practice can be smaller than that predicted by the models. The models suggested by Muravskii and Frydman [11], Muravskii [8,12], Osinov [7] are intended for eliminating this defect.

In the paper by Muravskii [8], Eq. (1) which represents the second rule is replaced by the following equation (with retention of remaining rules):

$$\tau = \tau_{\text{rev}} + \Phi(\gamma - \gamma_{\text{rev}}) \tag{4}$$

where the hysteresis function $\Phi(u)$ is introduced. Note that the function $\Phi(u)$ which corresponds to Eq. (1) has the form:

$$\Phi(u) = 2F_{\text{bb}}(u/2) \tag{5}$$

For materials with the symmetric behavior for positive and negative deformations, the function Φ should satisfy (similarly to the backbone curve) the condition of antisymmetry, i.e.

$$\Phi(-u) = -\Phi(u) \tag{6}$$

and contain as a parameter the absolute values of strain, γ_{bb}, at which the point (γ, τ) leaves the backbone curve because of unloading or reloading. This parameter remains without changes until the point (γ, τ) starts to move again along the backbone curve (according to the third or fourth rule) and afterwards abandons it at a new value γ_{bb}. For symmetric cycle deforming, the amplitude of deformations equals the parameter γ_{bb}, thus the hysteresis function should provide a defined value of damping ratio corresponding to amplitude γ_{bb} in a cyclic deforming process. An important property of the relationship (4) with the condition (6) of antisymmetry is that after an unloading (loading) with a following reloading (unloading) a point comes back in the reversal point where the unloading (loading) began (similarly to the classic Masing's model). In addition, the following requirements were imposed on the function Φ and its derivatives [8]:

$$\begin{aligned} &\text{(i)} \quad \Phi(0) = 0 \\ &\text{(ii)} \quad \Phi'(0) = F'_{bb}(0) \\ &\text{(iii)} \quad \Phi(2\gamma_{bb}) = 2\tau_{bb} \\ &\text{(iv)} \quad \Phi'(2\gamma_{bb}) = F'_{bb}(\gamma_{bb}) \\ &\text{(v)} \quad \Phi'(u) > 0 \text{ and } \Phi''(u) < 0 \; for \; 0 \leqslant u \leqslant 2\gamma_{bb} \end{aligned} \tag{7}$$

where $\tau_{bb} = F_{bb}(\gamma_{bb})$. In (v) the convexity of the corresponding curve is required. All these relationships correspond to the properties of the scaled backbone curve entering Eqs. (1) and (5). When using normalized values according to Eq. (3) γ and τ are changed to x and y, respectively, in above relationships; in this case initial derivative in (ii) equals one. Additional parameters which can be included into function Φ allow us to regulate its behavior in intermediate parts of the interval (0, $2\gamma_{bb}$) and thus to influence the form of hysteresis loops and damping properties of the model.

2. Simple examples of hysteresis functions

Calculations carried out for different kinds of hysteresis functions show that it is not necessary to fulfill all the requirements imposed in Eq. (7) on the hysteresis function. Further we omit conditions (ii), (iv) from Eq. (7) regarding derivatives. This allows us to treat not only the cases with the backbone curves with decreasing stiffness as of the type (2), but also backbone curves in the form of a straight line or with increasing stiffness. Consider the following hysteresis functions:

$$\Phi_1(\gamma) = \frac{2\gamma}{|\gamma|}\tau_{bb}\left(\frac{|\gamma|}{2\gamma_{bb}}\right)^{\alpha} \tag{8}$$

$$\Phi_2(\gamma) = \frac{\gamma}{\gamma_{bb}}\tau_{bb}\frac{1+\mu}{1+\mu|\gamma|/(2\gamma_{bb})} \tag{9}$$

These functions fulfill above conditions (with omitting (ii) and (iv)) and have an additional parameter, α or μ, which allows regulation of damping ratio. Damping ratio D is defined as

$$D = \frac{W}{2\pi\gamma_{bb}\tau_{bb}} \tag{10}$$

where W is the energy loss for period in a cyclic process with strain amplitude γ_{bb} and stress amplitude τ_{bb}. For a hysteresis function $\Phi(u)$, value of D can be represented in the form:

$$D = \frac{1}{\pi\gamma_{bb}\tau_{bb}}\left[\int_0^{2\gamma_{bb}} \Phi(u)\,u - 2\gamma_{bb}\tau_{bb}\right] \tag{11}$$

In the case of function (5), (2) which fulfills all the original Masing's rules we obtain from Eq. (11) damping ratio as the following function of amplitude γ_{bb}:

$$D = \frac{2}{\pi}\left[\frac{2(1+\gamma_{bb})}{\gamma_{bb}}\left(1 - \frac{\ln(1+\gamma_{bb})}{\gamma_{bb}}\right) - 1\right] \tag{12}$$

with maximum value $D_{max} = 2/\pi$ for $\gamma_{bb} \to \infty$. For the functions (8), damping ratio has the form:

$$D = \frac{2}{\pi}\frac{1-\alpha}{1+\alpha} \tag{13}$$

which results in

$$\alpha = \frac{1 - \pi D/2}{1 + \pi D/2} \tag{14}$$

For the function (9)

$$D = \frac{2}{\pi}\left[\frac{2(1+\mu)}{\mu^2}(\mu - \ln(1+\mu)) - 1\right] \tag{15}$$

Note that $D \to 2/\pi$ when $\mu \to \infty$. To express the parameter μ through D we introduce value ξ

$$\xi = \frac{\mu}{1+\mu} \tag{16}$$

and using Eq. (15) plot ξ vs. $r = \pi D/2$ (Fig. 1). This relationship allows the following good approximation at interval (0,1):

$$\xi \approx r + r(1-r)(1.98557 - 2.28055\,r + 1.8018\,r^2 - 0.623\,r^3) \tag{17}$$

with maximum error less than 0.0008 which is achieved at values r close to 1. Knowing ξ we find μ as $\xi/(1-\xi)$. Obviously, value of r should be less than one ($D < 2/\pi$). Clearly in above formulas arbitrary normalization for strains and stresses can be used; below the normalization (3) is applied in the case of the mechanical system having limited strength with substitution x, x_{bb}, y_{bb} instead of γ, γ_{bb}, τ_{bb}, respectively.

Consider how hysteresis functions (8) and (9) can approximate hysteretic loops corresponding to pure Masing's model with backbone curve (2) and hysteresis functions (5) having damping ratio (12). In Fig. 2, the hysteretic loops are shown for three values of $x_{bb} = 2$, 10, 30 with $D = 0.2241$, 0.4281, 0.5285, respectively. In Fig. 2, also the results are presented for the hysteresis function from paper by Muravskii [8], which satisfies all the requirements (7), with parameter R of that paper defined through parameter R^* (which corresponds to infinite amplitudes) as

$$R = \frac{1.3 + x_{bb}}{1 + x_{bb}} R^* \tag{18}$$

The results for the latter function and for the function (9) are almost not distinguishable in Fig. 2 from those for the pure Masing's model. The function (8) leads to some deviation from the model of Masing. It appears that function (9) can be used instead of more complicated function from the paper of Muravskii [8] in cases of the backbone curves of the type (2).

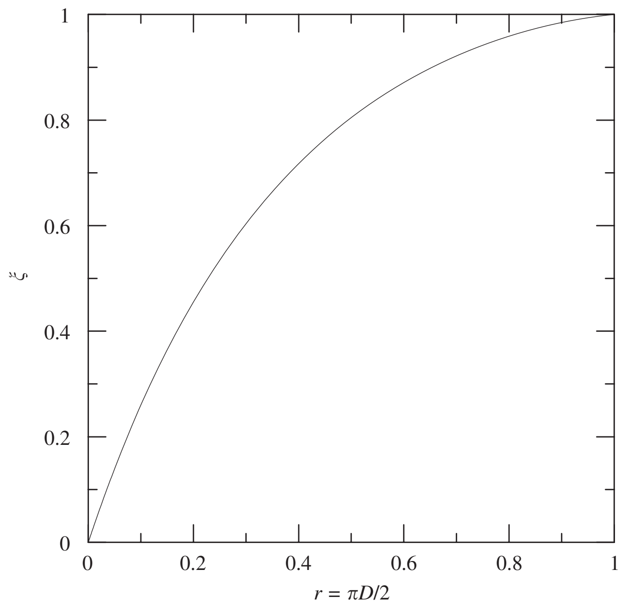

Fig. 1. Determination of parameter ξ for hysteresis function (9).

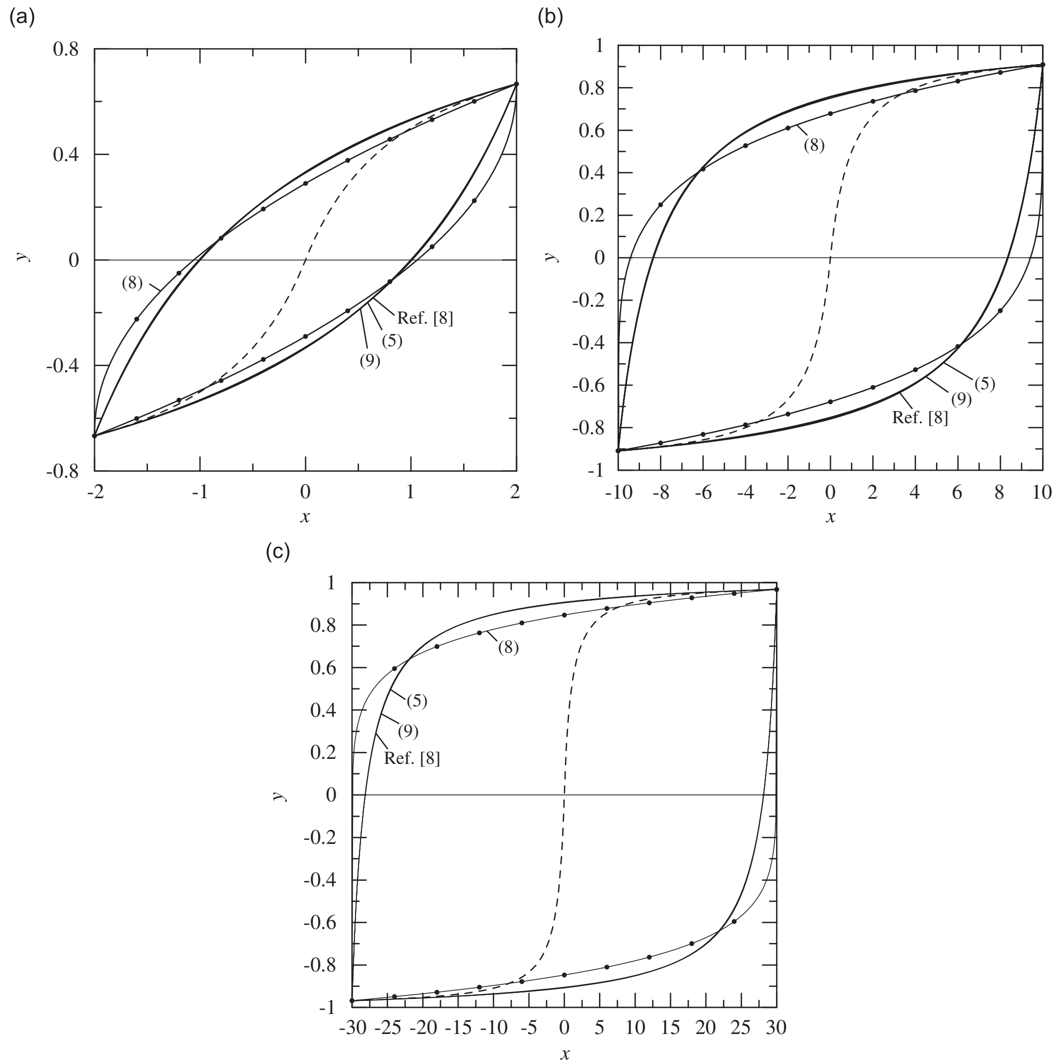

Fig. 2. Hysteretic loops for normalized strain amplitudes equal to 2 (a), 10 (b) and 30 (c); dashed line represents backbone curve; function (5) corresponds to pure Masing's model.

The two functions (8) and (9) represent some kind of extreme cases: with infinite and finite initial derivatives. Using these functions the following family of hysteresis functions can be constructed:

$$\Phi_\rho(\gamma) = \rho\Phi_1(\gamma) + (1-\rho)\Phi_2(\gamma) \tag{19}$$

where $0 \leqslant \rho \leqslant 1$ and the functions Φ_1, Φ_2 correspond to identical values of damping ratio D for a strain amplitude (the same value is inherent in the function Φ_ρ). The variety of hysteresis functions (19) makes it possible meeting different kinds of hysteretic behavior.

3. Dynamic examples for the case of backbone curve with limiting stiffness

Consider some dynamic problems for backbone curve (2). Let a mass m be attached to a spring which obeys the nonlinear hysteretic stress–strain relationship. We keep notations G_0, τ, γ for the stiffness, force, displacement, respectively (as the 'spring' one can image a shear rod which has square cross section with unit sides). Equation of motion has the form:

$$m\ddot{\gamma} + \tau = T \tag{20}$$

where T is an external force. Use of the normalized values x and y according to Eq. (3) and the non-dimensional time

$$\tilde{t} = t\sqrt{\frac{G_0}{m}} \tag{21}$$

leads to

$$\frac{d^2x}{d\tilde{t}^2} + y = \tilde{T} \tag{22}$$

where $\tilde{T} = T/\tau_u$. This equation was solved numerically (details are given in Appendix) for above hysteresis functions in the case of $T = 0$, $x(0) = 0$ and value of initial derivative $dx/d\tilde{t}$ equal to 5 (action of an instantaneous impulse); values of damping ratio are $D_{max} = 2/\pi$, 0.3, 0.1. In the first case along with functions (8) and (9) also function (5), (2) can be applied with corresponding Eq. (12) for damping ratio. This equation is used for functions (8) and (9) for current value x_{bb}, i.e. parameters α and μ are adjusted according to values of D from Eq. (12). The results of calculations are shown in Fig. 3. Plots for Masing's model and the model with function (9) ($D_{max} = 2/\pi$) are very close. Function (8) leads to a faster decay of oscillations and higher residual displacements. For this function loop areas for small oscillations with a non-zero mean value are greater than those corresponding to Masing's model and the model with function (9) (see Fig. 4), although the areas of loops symmetrical with respect to the point (0,0) are identical (Fig. 2). When applying function (19) with different value ρ, the corresponding plots will lie between plots for functions (8) and (9). In the case of the two other values of D_{max} only models with functions (8) and (9) are considered. In these cases the following equation for damping ratio is applied:

$$D = D_{max}\frac{x_{bb}}{1 + x_{bb}} \tag{23}$$

Note that the similar amplitude dependence has been recommended for soils in paper of Hardin and Drnevich [13]. The above examples show that the form of hysteretic loops can significantly influence dynamic response of mechanical systems having nearly identical dissipative properties for symmetric cyclic processes of deformation.

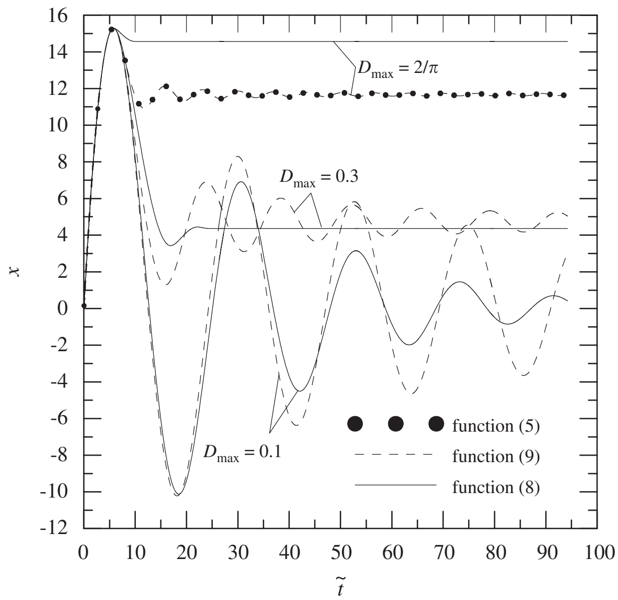

Fig. 3. Response of single-degree-of-freedom system with hyperbolic backbone curve on instantaneous impulse for different hysteresis functions.

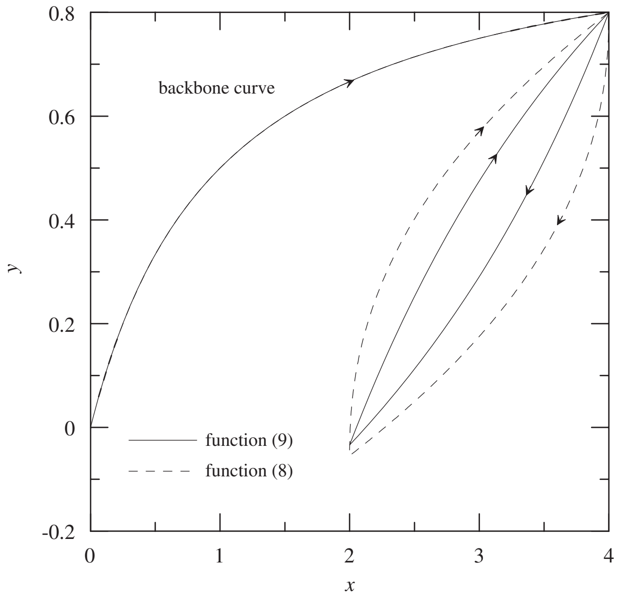

Fig. 4. Different areas of loops with non-zero mean strain for two different hysteresis functions having identical damping ratios corresponding to symmetrical loops (for $x_{bb} = 4$).

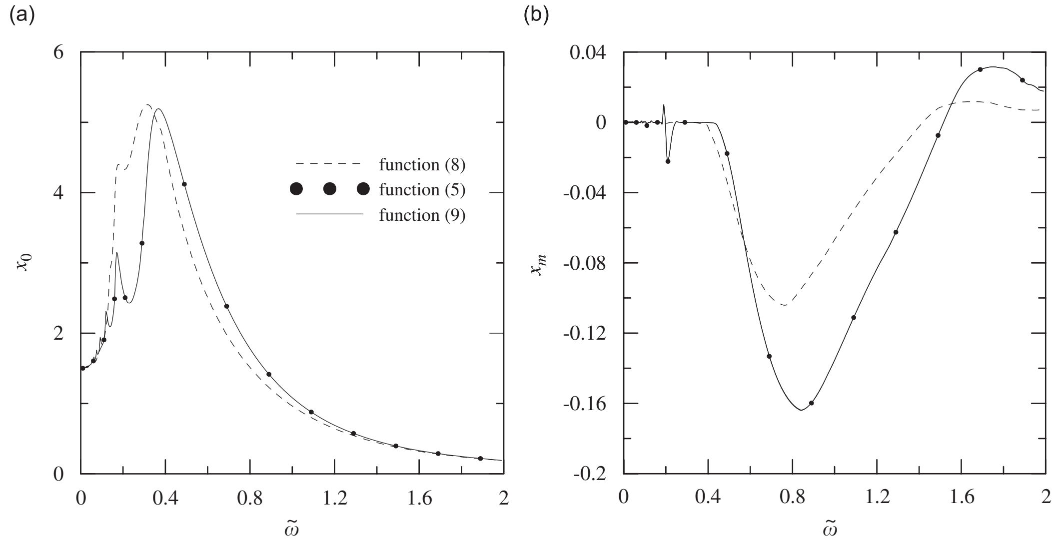

Fig. 5. Steady-state amplitudes (a) and mean displacements (b) vs. non-dimensional frequency for different hysteresis functions.

As the second example consider steady-state vibrations of hysteretic systems obeying Eq. (22) in the case of zero initial values of x and $dx/d\tilde{t}$ and $\tilde{T} = a \sin(\tilde{\omega}\tilde{t})$, where $\tilde{\omega}$ is a non-dimensional frequency. If ω is the cyclic frequency corresponding to t then $\tilde{\omega} = \omega t/\tilde{t}$. After large number of periods the process becomes practically steady state. Minimum and maximum displacements at the two last periods are determined and hence we found the amplitude of vibrations x_0 and the corresponding mean value x_m. For $a = 0.6$ amplitudes of vibrations are presented (Fig. 5a) vs. non-dimensional circle frequency $\tilde{\omega}$ for backbone curve (2) and three hysteresis functions (5), (8), (9) with $D_{max} = 2/\pi$ and the amplitude dependence for damping ratio according to Eq. (12). The mean value x_m for steady-state vibrations can deviate slightly from zero for some values of $\tilde{\omega}$

which is illustrated in Fig. 5b. Results corresponding to pure Masing's model and to the model using function (9) with the damping dependence (12) are identical.

4. Application of hysteresis functions in the case of linear backbone curve

Consider the backbone curve in the form

$$\tau = G_0\gamma \tag{24}$$

Using functions (8) and (9) we suppose that damping ratio does not depend on strain amplitude. In this case parameters α or μ also are independent of strain amplitude becoming together with damping ratio D the fixed characteristics of the given mechanical system. The hysteresis function can be represented in the form:

$$\Phi(\gamma) = G_0\gamma_{bb}\Phi_0(\gamma/\gamma_{bb}) \tag{25}$$

where the function $\Phi_0(x)$ is the hysteresis function for the values $G_0 = 1$, $\gamma_{bb} = 1$. In the case (8)

$$\Phi_0(u) = \frac{2u}{|u|}\left(\frac{|u|}{2}\right)^{\alpha} \tag{26}$$

In the case (9)

$$\Phi_0(u) = u\frac{1+\mu}{1+\mu|u|/2} \tag{27}$$

In the case (19)

$$\Phi_0(u) = \rho\frac{2u}{|u|}\left(\frac{|u|}{2}\right)^{\alpha} + (1-\rho)u\frac{1+\mu}{1+\mu|u|/2} \tag{28}$$

The property expressed in Eq. (25) leads to partial linearity of the considered system which was pointed out in paper by Muravskii [8]: for two strain histories, $\gamma_1(t)$ and $\gamma_2(t) = C\gamma_1(t)$, where C is a constant, the corresponding stresses will be in the same proportion, $\tau_2(t) = C\tau_1(t)$. In fact, for an initial stage when the point (γ,τ) moves along the linear backbone curve this statement is obvious. At the first unloading the amplitudes γ_{bb}, τ_{bb} will be in the same proportion as well as the corresponding hysteresis functions and stresses according to Eqs. (25) and (4). The relation will remain for subsequent reversal points, etc. This important property allows us to consider, for example, the response of the system on the unit instantaneous impulse and use the corresponding proportion for an arbitrary impulse value. Analogously, when studying steady-state vibrations under action of a harmonic force, it is enough to consider only unit force amplitude. Thus it is possible to compare in some cases the considered nonlinear system with linear one.

Further a one-degree-of-freedom system is addressed. Displacements and forces are considered now instead of strains and stresses. Using Eqs. (20) and (21) results in Eq. (22) where for normalization some value γ_r and the force $G_0\gamma_r$ are used

$$x = \frac{\gamma}{\gamma_r}, \quad y = \frac{\tau}{G_0\gamma_r}, \quad \tilde{T} = \frac{T}{G_0\gamma_r} \tag{29}$$

For these normalized values Eq. (24) changes to $y = x$, and Eq. (25) takes the form

$$\Phi(x) = x_{bb}\Phi_0(x/x_{bb}) \tag{30}$$

whereas Eqs. (26)–(28) remain valid. Consider the action of an instantaneous impulse with $T = 0$, $x(0) = 0$ and the value of initial derivative $dx/d\tilde{t}$ equal to 1. The reference displacement γ_r is taken as the unit displacement. In Fig. 6 the displacements for values of damping ratio $D = 0.1, 0.2$ are shown for the hysteresis functions (8), (9) or (26), (27). For comparison, the results corresponding to a linear model with viscose damping are represented in the figure (dashed lines). For this model

$$\tau = G_0\gamma + \delta\frac{d\gamma}{dt} = G_0\left(\gamma + 2\xi\frac{d\gamma}{d\tilde{t}}\right) \tag{31}$$

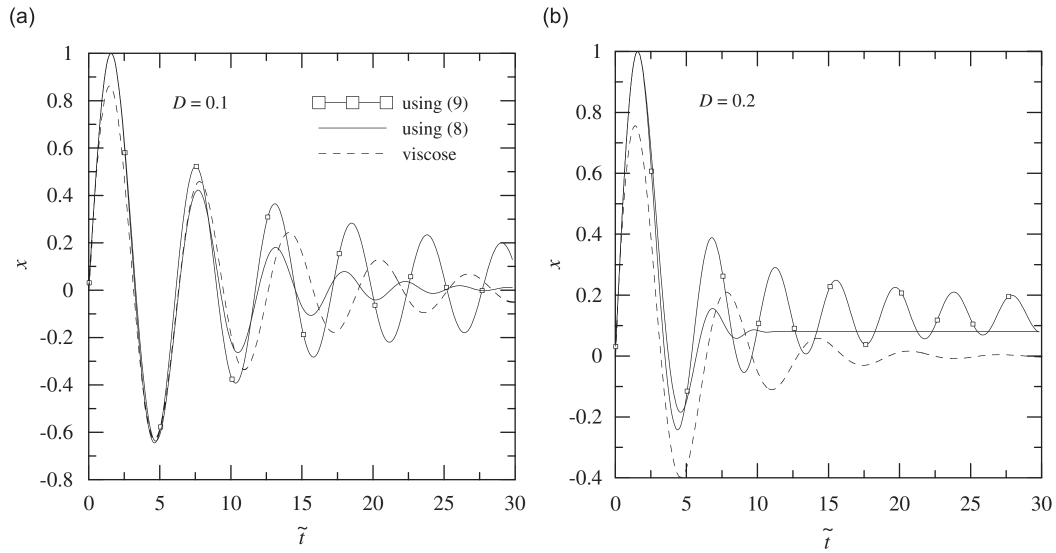

Fig. 6. Response of single-degree-of-freedom system with linear backbone curve on instantaneous impulse for hysteresis functions (8) and (9) and equivalent viscose model.

or

$$y = x + \delta \frac{dx}{dt} = x + 2\xi \frac{dx}{d\tilde{t}} \tag{32}$$

where the coefficient δ is a constant and

$$\xi = \frac{\delta}{2\sqrt{mG_0}} \tag{33}$$

In calculations the damping parameter ξ is taken equal to D which leads to the linear model which is 'equivalent' to original nonlinear hysteretic model. The results of calculations are rather close for the all models in the case of small damping. With increase of damping the discrepancies become noticeable. Note that for the hysteretic models the first maximum displacement does not depend on which hysteresis function is used, since the mechanical system behaves as the pure elastic system until the first reversal point is achieved. Differences in results for the considered hysteresis functions become apparent after first two limiting displacements. The model corresponding to function (9) has relatively slow decrease in amplitudes which was already explained when studying the system with the hyperbolic backbone curve. The response of the hysteretic system on impulse action is characterized by residual displacements which increase with increase in the damping ratio. The linear viscose system considered here for comparison has no residual displacement, and the damping leads to decrease in the first maximum of displacements.

As the second example for the case of linear backbone curve, steady-state vibration of the one-degree-of-freedom mechanical system under the action of an external force in the form

$$T = A \sin(\omega t) \tag{34}$$

is considered. Taking for normalization (29) $\gamma_r = A/G_0$ we obtain the unit normalized force amplitude, i.e., in Eq. (22) $\tilde{T} = \sin(\tilde{\omega}\tilde{t})$. The treatment used when studying the system with hyperbolic backbone curve is applied. As above the results are presented for the same two hysteresis functions and the equivalent viscose model for two values of damping ratio $D = 0.1$ and 0.2 (Figs. 7 and 8).

The deviation of the mean displacement x_m from zero becomes noticeable with increase of damping ratio. Dashed lines represent results corresponding to equivalent viscose model for which

$$x_0 = \frac{1}{\sqrt{(1-\tilde{\omega}^2)^2 + 4D^2\tilde{\omega}^2}} \tag{35}$$

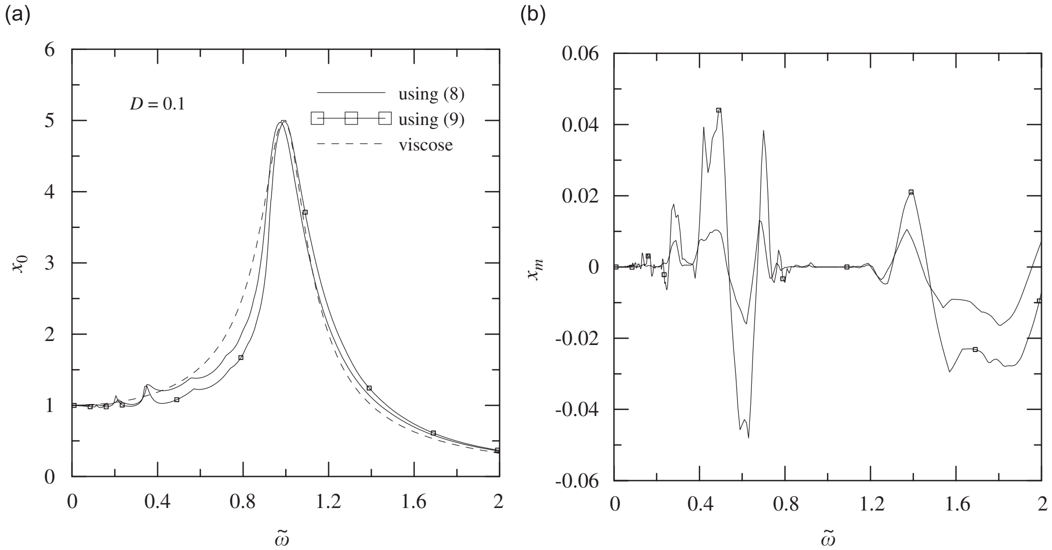

Fig. 7. Steady-state amplitudes (a) and mean displacements (b) vs. non-dimensional frequency for two hysteresis functions and equivalent viscose model; linear backbone curve; $D = 0.1$.

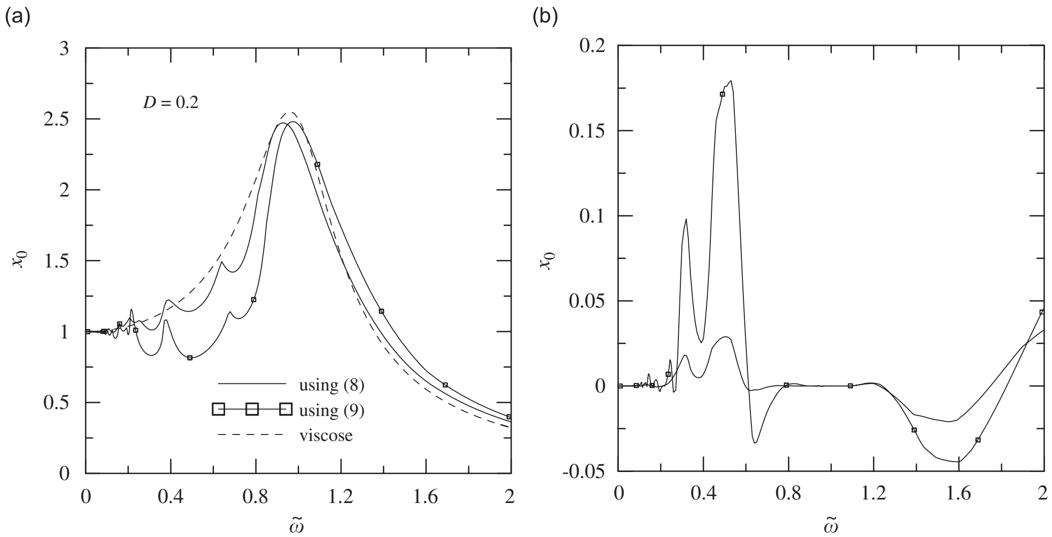

Fig. 8. Steady-state amplitudes (a) and mean displacements (b) vs. non-dimensional frequency for two hysteresis functions and equivalent viscose model; linear backbone curve; $D = 0.2$.

5. Application of hysteresis functions in the case of backbone curve with increasing stiffness

The new treatment that neglects the requirements (ii) and (iv) in Eq. (7) can be effectively applied also in the case of a backbone curve with increasing stiffness, e.g., of the following form:

$$\tau = G_0(\gamma + \lambda\gamma^3) \tag{36}$$

where G_0 is the initial stiffness and λ is a positive parameter. After normalization of the type (29) Eq. (36) acquires the form

$$y = x + \tilde{\lambda}x^3 \tag{37}$$

with $\tilde{\lambda} = \lambda\gamma_r^2$. The reference strain (displacement) γ_r can be taken equal to 1. When solving dynamic problems, the damping ratio dependence on the strain amplitude should be given. For following examples we assume that damping ratio does not depend on the vibration amplitude. Below vibrations of a one-degree-of-freedom system is considered using Eqs. (20)–(22), (29), (37). Results of calculations for the case $T = 0$, $x(0) = 0$ and the value of initial derivative $\mathrm{d}x/\mathrm{d}\tilde{t}$ equal to 5 and $\tilde{\lambda} = 0.5$ are presented in Fig. 9 for two hysteresis functions (8) and (9). As above the hysteresis function (8) leads to faster decrease in amplitudes in comparison to the function (9). The residual displacements increase with growth of the damping ratio. The case of steady-state vibrations under action of an external force (34) is studied for two hysteresis functions (8) and (9) for the values of damping ratio $D = 0.1$ and 0.2; the reference displacement γ_r is taken equal to 1. In the examples

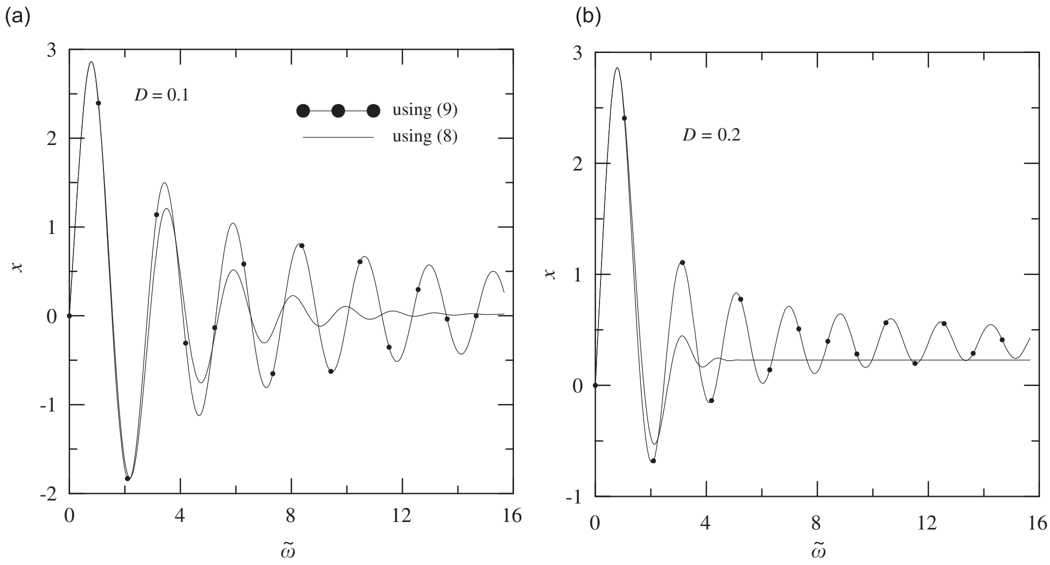

Fig. 9. Response of single-degree-of-freedom system with backbone curve (37) ($\tilde{\lambda} = 0.5$) on instantaneous impulse for two hysteresis functions and for damping ratio $D = 0.1$ (a) and $D = 0.2$ (b).

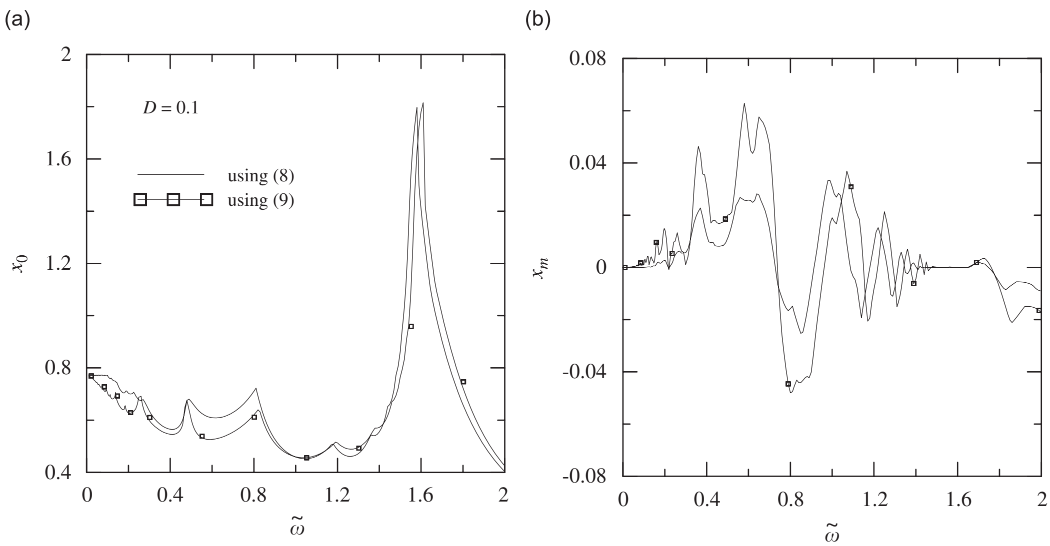

Fig. 10. Steady-state amplitudes (a) and mean displacements (b) versus non-dimensional frequency for two hysteresis functions; backbone curve (37) with $\tilde{\lambda} = 0.5$ and $D = 0.1$.

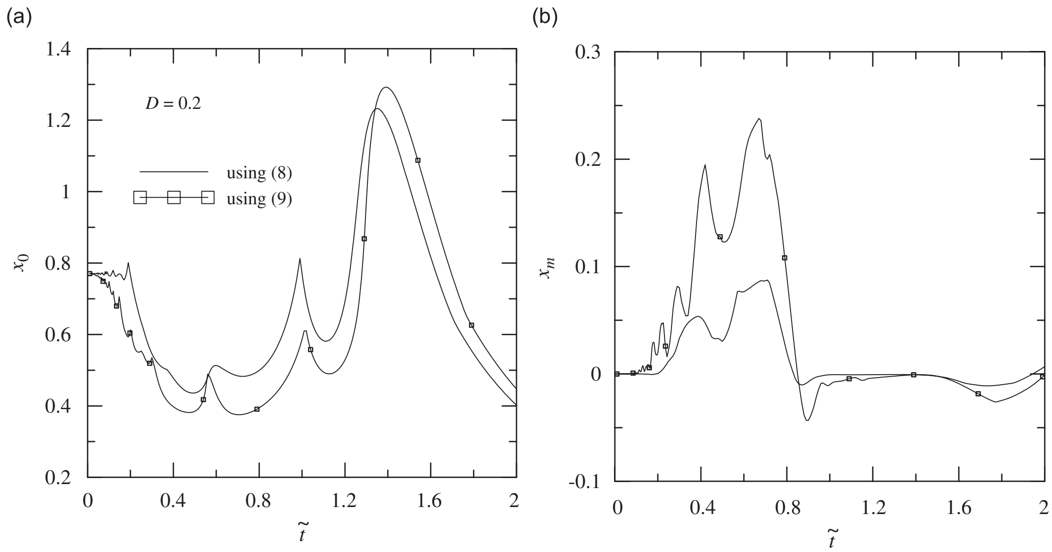

Fig. 11. Steady-state amplitudes (a) and mean displacements (b) versus non-dimensional frequency for two hysteresis functions; backbone curve (37) with $\tilde{\lambda} = 0.5$ and $D = 0.2$.

non-dimensional force amplitude $\tilde{T}$ is equal to 1. The corresponding results are shown in Figs. 10 and 11. As in the case of linear backbone curve the middle point x_m for steady-state vibrations is shifted from zero for some values of frequency.

6. Concluding remarks

As is shown in the paper the requirements imposed on the hysteresis functions in the previous publication [8] of the author can be loosened and thus simpler hysteresis functions can be applied for description of the hysteretic behavior of materials. Two simple basic functions and a family generated by these functions are proposed and their application for three cases of backbone curves: (a) with limiting stress, (b) linear backbone curve, and (c) backbone curve with increasing stiffness has been demonstrated. Vibrations of a one-degree-of-freedom system under action of an instantaneous impulse and a harmonic force have been studied for the three cases of the backbone curve. Results of calculations for different hysteresis functions show that the form of a hysteretic loop, corresponding to symmetrical cycle deforming, influences significantly the dynamic behavior of a mechanical system. In fact, two hysteresis functions with finite and infinite initial derivative for which symmetrical loops have equal areas (damping ratios) but different forms, lead to sharply different levels of damping in transient vibrations.

Appendix

Below an example of program codes in Pascal is presented which relates to the vibration problem for a one-degree-of-freedom system. The function (8) is applied using the normalized values (3) for the backbone curve (2), $D_{\max} = 0.3$, the damping ratio dependence on amplitude is in accordance with Eq. (23). The zero initial displacement x and initial derivative $x1 = 5$ are taken into account, and the external force is equal to zero. The method of mean constant accelerations with iterations is chosen for solution of the corresponding differential equation. Difficulties of programming, arisen due to necessity of matching the third and fourth Masing's rules, are overcome using the array of sequential reversal points. Comments are given in braces.

```
program new_masing;
label 1;
var i, nst:longint;
```

```
itr,nitr,np,n0,ii:integer;
x,x1,x2,xx,xx1,xx2,t,yy,dr,dam,al, mean,xr,yr,t1,st,xbb:extended;
fil:text;
nam:string[86];
xtu,ytu:array[−1..200] of extended;
{array of reversal points in plane (x,y); when returning to the backbone curve the account begins from 0}

function fbb(xx:extended):extended;
{backbone curve}
begin fbb: = xx/(1 + abs(xx)) {hyperbolic}
  end;

function fmu(xx:extended):extended;
{the hysteresis function (8) for normalized values}
begin if xx = 0 then fmu: = 0 else begin
  fmu: = 2*xx/abs(xx)*fbb(xbb)*exp(al*ln(abs(xx)/2/xbb));
    end
end;

function force(t:extended):extended;
{the function in right-hand side of Eq. (22)}
begin
force: = 0
end;

procedure ini;
{initial values for integration}
{displacement x, velocity x1,acceleration x2}
begin x: = 0;x1: = 5;x2: = force(0)
end;

function yi(xx,xt,yt:extended;ui:integer):extended;
{force (stress) yi versus displacement (strain) xx;(xt,yt) is last reversal point; parameter ui = 0 corresponds to point (xx,yi) location on backbone curve}
begin if ui = 0 then yi: = fbb(xx) else begin
yi: = yt + fmu(-xt + xx) {see Eq. (4)}
                          end
end;

begin {main}
nitr: = 3; {number of iterations at each time step}
dam: = 0.3; { Dmax corresponds to the infinite amplitude}
nam: = '200-imp5-alf.dat'; {file for output}
assign(fil,nam);rewrite(fil);
writeln(fil,' D = ',' ',dam:9:7,' ','imp = 5');
n0: = 200; {step number at period}
st: = 2*pi/n0; {time step}
np: =  10; {number of periods}
nst: = np*n0; {number of time steps}
ini; {initial values}
ii: = 0; {ii is the current number of the return point}
xtu[-1]: = 1000;ytu[-1]: = 1000; {artificial return point}
for i: = 1 to nst do begin t: = i*st;xx2: = x2;
{previously the acceleration at a current time step end is set equal to the acceleration at the step start}
for itr: = 1 to nitr do
```

```
begin
mean: = (x2 + xx2)/2; {mean acceleration}
xx1: = x1 + mean*st; {velocity at the time step end}
xx: = x + x1*st + mean*st*st/2; {displacement at the time step end}
yy: = yi(xx,xtu[ii],ytu[ii],ii);
if x1*xx1 < 0 then
   begin
      {we control possibility of reversal point at the step}
      t1: = st*abs(x1)/(abs(x1) + abs(xx1));
      {t1 is the local time of reversal;(xr,yr) is point of reversal}
      xr: = x + 0.5*t1*x1;yr: = yi(xr,xtu[ii],ytu[ii],ii);
   if ii = 0 then begin
{abandoning backbone curve with determination of amplitude xbb, damping ratio dr and parameter al (α) }
xbb: = abs(xr);
dr: = dam*xbb/(1 + xbb);
al: = (1-pi*dr/2)/(1 + pi*dr/2);
xtu[0]: = -xr;ytu[0]: = -yr {for possible future arrival to another branch of backbone curve}
               end;
yy: = yi(xx,xr,yr,ii + 1);
if itr = nitr then begin
   xtu[ii + 1]: = xr;ytu[ii + 1]: = yr;
   ii: = ii + 1;
                  end end;
if (x-xtu[ii-1])*(xx-xtu[ii-1]) < = 0
{checking arrival to the previous reversal point}
then begin
1: if ii-1 = 0 then yy: = fbb(xx) {see the third Masing's rule }
      else
yy: = yi(xx,xtu[ii-2],ytu[ii-2],ii-2); {see the fourth Masing's rule }
if ii = 1 then begin ii: = 0 end else ii: = ii-2;
if (x-xtu[ii-1])*(xx-xtu[ii-1]) < = 0
{possibility of more than one previous reversal point at the same step}
then goto 1
   end;
xx2: = force(t)-yy; {Eq. (22)}
writeln(ii,' ', i,' ',itr,' ',xx2,' ',xx)
   end;
   x2: = xx2;x1: = xx1;x: = xx; {preparation to next step}
writeln(fil,ii,' ',t:8:6,' ',' ',x:13:10,' ',x1:13:10,' ',yy:10:6)
   end;
   close(fil);
end.
```

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